Fractal Dimension Results for Lévy and Lévy-Type Processes

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1. Introduction: Hausdorff dimension of the image

If $\{X(t), t \geq 0\}$ is a Lévy process in $\mathbb{R}^d$, there have been a lot of results on its fractal properties of the random sets. Our starting point is the following theorem.

Theorem 1.1 (Blumenthal and Getoor, 1960)

If $\{X(t), t \geq 0\}$ is a strictly α-stable Lévy process in $\mathbb{R}^d$ and $\alpha \in (0, 2)$. Then for every Borel set $E \subset \mathbb{R}_+$,

$$\dim_{\rm H} X(E) = \min\{d, \alpha \dim_{\rm H} E\} \quad \text{a.s.}$$

In this theorem, the null probability event where the equality does not hold depends on E.

Introduction: Hausdorff dimension of the inverse image

Theorem 1.2 (Hawkes, 1971)

Let $X = \{X(t), t \geq 0\}$ be an α-stable Lévy process in $\mathbb{R}^d$ and let $F \subset \mathbb{R}^d$ be a fixed Borel set. Then

$$\left\| \dim_{\mathrm{H}} X^{-1}(F) \right\|_{L^\infty(\mathbb{P})} = 1 - \frac{1}{\alpha} + \frac{1}{\alpha} \dim_{\mathrm{H}} F,$$

where $\|Z\|_{L^\infty(\mathbb{P})} = \sup \left\{ \theta : \mathbb{P}(Z \geq \theta) > 0 \right\}$.

Questions on "uniform dimension results"

Let $X = \{X(t), t \geq 0\}$ be an α-stable Lévy process in $\mathbb{R}^d$.

(i) Is $\mathbb{P}\{\dim_{\rm H} X(E) = \min\{d, \alpha \dim_{\rm H} E\},\ \forall\, E \subset \mathcal{B}(\mathbb{R}_+)\} = 1$?

(ii) Is

$$\mathbb{P}\Big\{\dim_{\rm H} X^{-1}(F) = \frac{\alpha + \dim_{\rm H} F - d}{\alpha},\ \forall\, F \subset \mathcal{B}(\mathbb{R}^d)\Big\} = 1?$$

The first such results are due to R. Kaufman, who gave an affirmative answer to (i) in 1968 for planar Brownian motion, and to (ii) for Brownian motion in $\mathbb{R}$ in 1985.

Remarks:

- (i) does not hold if $\alpha > d = 1$. For example, take $E = X^{-1}(\{0\})$.
- (ii) does not hold if $\alpha < d$. For example, take $F = X([0, 1])$.

Question (i) was answered by Hawkes and Pruitt (1974) for stable Lévy processes, while (ii) was answered recently by Song, X. and Yang (2018).

Theorem 1.3 [Hawkes and Pruitt (1974)]

If $\{X(t), t \geq 0\}$ is a strictly α-stable Lévy process with values in $\mathbb{R}^d$ and $\alpha \leq d$. Then almost surely

$$\dim_{\rm H} X(E) = \alpha \dim_{\rm H} E, \ \ \forall E \in \mathcal{B}(\mathbb{R}_+).$$

Refinements and packing dimension analog of Theorem 1.3 were proved by Perkins and Taylor (1987).

The validity of (ii) in the case $\alpha = 2$ (X is a Brownian motion) and $d = 1$ is due to Kaufman (1985): a.s.

$$\dim_{H} B^{-1}(F) = \frac{1}{2} + \frac{1}{2}\dim_{H} F, \quad \forall F \in \mathcal{B}(\mathbb{R}).$$

His proof relies on the uniform modulus of continuity of Brownian motion as well as the Hölder continuity of the Brownian local time in the time variable.

Another interesting result is due to Barlow, Perkins and Taylor (1987) for the special case of $F = \{x\}$: If $1 < \alpha \leq 2$ then a.s.,

$$\dim_{H} X^{-1}(x) = 1 - \frac{1}{\alpha}, \quad \forall\, x \in \mathbb{R}.$$

Moreover, Barlow, *et al* (1987) provided a geometric construction for the local times of X.

2. Uniform dimension result for the images

Let $X = \{X(t), t \geq 0, \mathbb{P}^x\}$ be a time-homogeneous Markov processes with values in $\mathbb{R}^d$. We assume it has the strong Markov property and its transition probability $P(t, x, A) := \mathbb{P}^x(X(t) \in A)$ satisfies the following conditions:

(A1) There is a constant $H > 0$ such that for any $\gamma \in (0, H)$, there exist constants $C > 0$, $\eta > 0$ and $t_0 \in (0, 1)$ such that for all $x \in \mathbb{R}^d$ and $0 < t \leq t_0$,

$$\mathbb{P}^x\Big\{ \sup_{0 \leq s \leq t} |X(s) - x| \geq t^\gamma \Big\} \leq Ct^\eta. \tag{1}$$

(A2) For any positive numbers ε, ζ and T, there exist positive constants C_1, C_2, and $r_0 \leq 1$ such that for all $0 < r \leq r_0$, $x, y \in \mathbb{R}^d$ with $|y - x| \leq r$, and all $0 < t \leq T$,

$$P(t, y, B(x, r)) \geq C_1 \min\left\{1, \left(\frac{r}{t^{H-\zeta}}\right)^{d+\varepsilon}\right\}; \qquad (2)$$

and

$$P(t, x, B(x, r)) \leq C_2 \min\left\{1, \left(\frac{r}{t^{H+\zeta}}\right)^{d-\varepsilon}\right\}. \qquad (3)$$

Theorem 2.1 [Sun, X., Xu and Zhai (2018)]

Let $X = \{X(t), t \in \mathbb{R}_+, \mathbb{P}^x\}$ be a time homogeneous Markov process in $\mathbb{R}^d$ and satisfies Conditions (A1) and (A2). If $1 \le Hd$, then for all $x \in \mathbb{R}^d$,

$$\mathbb{P}^x\Big\{\dim_{\rm H} X(E) = \frac{1}{H}\dim_{\rm H} E, \quad \forall\, E \subseteq \mathcal{B}(\mathbb{R}_+)\Big\} = 1. \tag{4}$$

The equality still holds if $\dim_{\rm H}$ is replaced by packing dimension.

They showed that this theorem is applicable to many Lévy and Lévy type processes.

Proof of the upper bound: Covering Lemma 1

In order to prove $\dim_{\rm H} X(E) \le \frac{1}{H}\dim_{\rm H} E$ for all E, they first prove the following lemma which extends Covering Principle 1 in Pruitt (1975).

Lemma 2.1

Let $\{t_n, n \ge 1\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} t_n^p < \infty$ for some $p > 0$, and let $\mathcal{C}_n$ be a class of N_n intervals in $\mathbb{R}_+$ of length t_n with $\log N_n = O(1)|\log t_n|$. If there is a sequence $\{\theta_n\}$ of positive numbers such that

$$\mathbb{P}^x\Big\{ \sup_{0\le s\le t_n} |X(s) - x| \ge \theta_n \Big\} \le C_3 t_n^{\delta}, \tag{5}$$

where C_3 and δ are some positive constants, then there exists a positive integer C_4, depending on p and δ only, such that, $\mathbb{P}^x$-a.s. for n large enough, $X(I)$ can be covered by C_4 balls of radius θ_n whenever $I \in \mathcal{C}_n$.

It can be proved that (5) holds under (A1). This is enough for proving $\dim_{\rm H} X(E) \le \frac{1}{H} \dim_{\rm H} E$ for all E.

Proof of the lower bound: Covering Lemma 2

The following is an extension of the Covering Principle 2 in Pruitt (1975).

Lemma 2.2

Let $\{r_n, n \geq 1\}$ be a sequence of positive numbers with $\sum_{n=1}^{\infty} r_n^p < \infty$ for some $p > 0$, and let $\mathcal{D}_n$ be a class of N_n balls of diameter r_n in $\mathbb{R}^d$ with $\log N_n = O(1)|\log r_n|$.
If, for every constant $T > 0$, there exists a sequence $\{t_n\}$ of positive numbers and constants C and $\delta > 0$ such that

$$\mathbb{P}^x\Big\{ \inf_{t_n \leq s \leq T} |X(s) - x| \leq r_n \Big\} \leq C r_n^{\delta}, \quad \forall\, x \in \mathbb{R}^d, \tag{6}$$

then there exists a constant C_5, depending on p and δ only, such that, with $\mathbb{P}^x$-probability one, for n large enough, $X^{-1}(B) \cap [0, T]$ can be covered by at most C_5 intervals of length t_n, whenever $B \in \mathcal{D}_n$.

3. Uniform dimension result for the inverse images

The following is an answer to (ii) for stable Lévy processes.

Theorem 3.1 [Song, X. and Yang (2018)]

Let X be a real-valued strictly α-stable Lévy process in $\mathbb{R}$ with $1 < \alpha \le 2$. Then a.s.,

$$\dim_{\rm H} X^{-1}(F) = 1 - \frac{1}{\alpha} + \frac{1}{\alpha} \dim_{\rm H} F, \quad \forall F \in \mathcal{B}(\mathbb{R}).$$

The proof is divided into two parts:

- Upper bound: a.s.

$$\dim_{\rm H} X^{-1}(F) \le 1 - \frac{1}{\alpha} + \frac{1}{\alpha}\dim_{\rm H} F, \quad \forall F \in \mathcal{B}(\mathbb{R}).$$

- Lower bound: a.s.

$$\dim_{\rm H} X^{-1}(F) \ge 1 - \frac{1}{\alpha} + \frac{1}{\alpha}\dim_{\rm H} F, \quad \forall F \in \mathcal{B}(\mathbb{R}).$$

Proof of the upper bound: Covering Lemma 3

The main tool for proving the upper bound is the following new covering lemma, which relies on results of Port (1967) and Port and Stone (1971) for the first hitting time of an interval by the process X.

Let $\mathcal{U}_n$ be the family of dyadic intervals of length 2^{-n} and let $\mathcal{V}_n$ be any partition of $\mathbb{R}_+$ with intervals of length $2^{-n\alpha}$.

Covering Lemma 3

Assume $1 \le \alpha \le 2$ and let $\delta > \alpha - 1$ and $T > 0$. $\mathbb{P}^x$-a.s., for all n large enough and $U \in \mathcal{U}_n$,

$X^{-1}(U) \cap [0, T]$ can be covered by $2^{n\delta+1}$ intervals from $\mathcal{V}_n$.

Proof. Let $U = (z - \frac{2^{-n}}{2}, z + \frac{2^{-n}}{2}) \in \mathcal{U}_n$.
Let $\tau_0 = 0$ and, for all $k \geq 1$, define

$$\tau_k = \inf\left\{s > \tau_{k-1} + 2^{-n\alpha} : |X(s) - z| < \frac{2^{-n}}{2}\right\}$$

with $\inf \emptyset = \infty$. Then

$$X^{-1}(U) \subset \bigcup_{i=0}^{\infty} [\tau_i, \tau_i + 2^{-n\alpha}],$$

which implies that, for any $T > 0$,

$$\{\tau_k > T\} \subset \left\{X^{-1}(U) \cap [0, T] \subset \bigcup_{i=0}^{k-1} [\tau_i, \tau_i + 2^{-n\alpha}]\right\}.$$

Therefore,

$$\left\{X^{-1}(U) \cap [0, T] \text{cannot be covered by } k \text{ intervals of length } 2^{-n\alpha}\right\}$$
$$\subset \left\{\tau_k \le T\right\}.$$

By the strong Markov property, we have

$$\begin{aligned}
\mathbb{P}^x(\tau_k \le T) &= \mathbb{P}^x(\tau_k \le T | \tau_{k-1} \le T)\mathbb{P}^x(\tau_{k-1} \le T) \\
&\le \sup_{y \in \overline{U}} \mathbb{P}^y\Big(\inf_{2^{-n\alpha} \le s \le T} |X(s) - z| \le 2^{-n}/2\Big)\mathbb{P}^x(\tau_{k-1} \le T) \\
&\le \sup_{y \in \overline{U}} \mathbb{P}^y\Big(\inf_{2^{-n\alpha} \le s \le T} |X(s) - y| \le 2^{-n}\Big)\mathbb{P}^x(\tau_{k-1} \le T) \\
&= p_n \cdot \mathbb{P}^x(\tau_{k-1} \le T),
\end{aligned}$$

where

$$p_n = \mathbb{P}^0\Big(\inf_{2^{-n\alpha} \le s \le T} |X(s)| \le 2^{-n}\Big).$$

By induction, we obtain

$$\mathbb{P}^x(\tau_k \le T) \le p_n^k.$$

By the independence of increments and the fact that $X(1)$ is supported on $\mathbb{R}$,

$$\begin{aligned} 1 - p_n &= \mathbb{P}^0\Big(\inf_{1 \le s \le T2^{n\alpha}} |X(s)| > 1\Big) \\ &\ge \mathbb{P}^0\Big(2 \le X(1) \le 3, \inf\{t \ge 1 : X(t) - X(1) \in [-4, -1]\} \ge T2^{n\alpha}\Big) \\ &\ge c\, \mathbb{P}^0(\mathcal{T}_{[-4,-1]} \ge T2^{n\alpha}), \end{aligned}$$

where $\mathcal{T}_{[-4,-1]}$ is the first hitting time of $[-4, -1]$ by X.

It follows from Port and Stone (1971) that

$$1 - p_n \geq c_T 2^{-n(\alpha-1)}.$$

Let $K \geq 1$ be fixed. For any $n \geq 1$, define the event

$$A_n^\delta = \Big\{ \exists U \in \mathcal{U}_n \cap [-K, K], \text{ s.t. } X^{-1}(U) \cap [0, T] \text{ cannot be covered by } 2^{n\delta} \text{ intervals of length } 2^{-n\alpha} \Big\}.$$

Then

$$\sum_{n=1}^{\infty} \mathbb{P}^x(A_n^\delta) \le \sum_{n=1}^{\infty} \sharp\{U \in \mathcal{U}_n : U \cap [-K, K] \neq \emptyset\}(p_n)^{2^{n\delta}}$$
$$\le 2K \sum_{n=1}^{\infty} 2^n\big(1 - c_T 2^{-n(\alpha-1)}\big)^{2^{n\delta}}$$
$$\le 2K \sum_{n=1}^{\infty} \exp\Big(n(\ln 2) - c_T\, 2^{n(\delta-\alpha+1)}\Big) < \infty,$$

because $\delta > \alpha - 1$. The lemma follows from the Borel-Cantelli Lemma.

Proof of the lower bound: local times

For proving the lower bound, we make use of Covering Lemma 1 and the following result of Perkins (1986) on the local times of stable Lévy processes:

$$\limsup_{r\to 0} \sup_{\substack{|s-t|<r \\ 0\le s<t\le 1}} \frac{L^*([s,t])}{r^{1-\frac{1}{\alpha}}(\log 1/r)^{\frac{1}{\alpha}}} = C_6, \quad \mathbb{P}^x\text{-a.s.}$$

wheren $L^*([s,t]) = \sup_{x\in\mathbb{R}}(L_t^x - L_s^x)$ is the maximum local time of X on $[s,t]$ and C_6 is a positive and finite constant.

Extension of Theorem 3.1

In order to prove similar results for more general Lévy processes or Lévy-type processes, we need to study the asymptotic properties of the first hitting times and the local times of these Markov processes.

Grzywny and Ryznar (2017) studied first hitting times of an interval by a symmetric Lévy process with regularly varying exponents.

Recently, Park, X. and Yang (2019) have extended Theorem 3.1 to more general Lévy processes.

Thank you !